~~Appendix C~~Attachment 9
Balancing the Type I and Type II Error Probabilities of the Truncated Z Test Statistic

This appendix describes a the methodology for balancing the error probabilities when the Truncated Z statistic, described in Appendix A, is used for performance measure parity testing. There are four key elements of the statistical testing process:

1. the null hypothesis, H_0, that parity exists between ILEC and CLEC services
2. the alternative hypothesis, H_a, that the ILEC is giving better service to its own customers
3. the Truncated Z test statistic, Z^T, and
4. a critical value, *c*

The decision rule[1] is

- If $Z^T < c$ then accept H_a.
- If $Z^T \geq c$ then accept H_0.

There are two types of error possible when using such a decision rule:

Type I Error: Deciding favoritism exists when there is, in fact, no favoritism.
Type II Error: Deciding parity exists when there is, in fact, favoritism.

The probabilities of each type of each are:

Type I Error: $\alpha = P(Z^T < c \mid H_0)$.
Type II Error: $\beta = P(Z^T \geq c \mid H_a)$.

In what follows, we show how to find a balancing critical value, c_B, so that $\alpha = \beta$.

General Methodology

The general form of the test statistic that is being used is

[1] This decision rule assumes that the smaller a performance measure is, the better the service. If the opposite is true, then reverse the decision rule.

$$z_0 = \frac{\hat{T} - E(\hat{T}|H_0)}{SE(\hat{T}|H_0)}, \tag{C.1}$$

where

$\hat{T}$ is an estimator that is (approximately) normally distributed,

$E(\hat{T}|H_0)$ is the expected value (mean) of $\hat{T}$ under the null hypothesis, and

$SE(\hat{T}|H_0)$ is the standard error of $\hat{T}$ under the null hypothesis.

Thus, under the null hypothesis, z_0 follows a standard normal distribution. However, this is not true under the alternative hypothesis. In this case,

$$z_a = \frac{\hat{T} - E(\hat{T}|H_a)}{SE(\hat{T}|H_a)}$$

has a standard normal distribution. Here

$E(\hat{T}|H_a)$ is the expected value (mean) of $\hat{T}$ under the alternative hypothesis, and

$SE(\hat{T}|H_a)$ is the standard error of $\hat{T}$ under the alternative hypothesis.

Notice that

$$\begin{aligned} \beta &= P(z_0 > c \,|\, H_a) \\ &= P\left(z_a > \frac{c\,SE(\hat{T}|H_0) + E(\hat{T}|H_0) - E(\hat{T}|H_a)}{SE(\hat{T}|H_a)} \right) \end{aligned} \tag{C.2}$$

and recall that for a standard normal random variable z and a constant b, $P(z < b) = P(z > -b)$. Thus,

$$\alpha = P(z_0 < c) = P(z_0 > -c) \tag{C.3}$$

Since we want $\alpha = \beta$, the right hand sides of (C.2)~~(C.2)(C.2)(C.2)~~ and (C.3)~~(C.3)(C.3)(C.3)~~ represent the same area under the standard normal density. Therefore, it must be the case that

$$-c = \frac{c\,SE(\hat{T}|H_0) + E(\hat{T}|H_0) - E(\hat{T}|H_a)}{SE(\hat{T}|H_a)}.$$

Solving this for *c* give the general formula for a balancing critical value:

$$c_B = \frac{E(\hat{T}|H_a) - E(\hat{T}|H_0)}{SE(\hat{T}|H_a) + SE(\hat{T}|H_0)} \tag{C.4}$$

The Balancing Critical Value of the Truncated Z

In Appendix A, the Truncated Z statistic is defined as

$$Z^T = \frac{\sum_j W_j Z_j^* - \sum_j W_j E(Z_j^*|H_0)}{\sqrt{\sum_j W_j^2 Var(Z_j^*|H_0)}}$$

In terms of equation (C.1)~~(C.1)~~~~(C.1)~~~~(C.1)~~ we have

$$\hat{T} = \sum_j W_j Z_j^*$$

$$E(\hat{T}|H_0) = \sum_j W_j E(Z_j^*|H_0)$$

$$SE(\hat{T}|H_0) = \sqrt{\sum_j W_j^2 Var(Z_j^*|H_0)}$$

To compute the balancing critical value (C.4)~~(C.4)~~~~(C.4)~~~~(C.4)~~, we also need $E(\hat{T}|H_a)$ and $SE(\hat{T}|H_a)$. These values are determined by

$$E(\hat{T}|H_a) = \sum_j W_j E(Z_j^*|H_a), \text{ and}$$

$$SE(\hat{T}|H_a) = \sqrt{\sum_j W_j^2 \, var(Z_j^*|H_a)}.$$

In which case equation (C.4)~~(C.4)~~~~(C.4)~~~~(C.4)~~ gives

$$c_B = \frac{\sum_j W_j E(Z_j^*|H_a) - \sum_j W_j E(Z_j^*|H_0)}{\sqrt{\sum_j W_j^2 \, var(Z_j^*|H_a)} + \sqrt{\sum_j W_j^2 \, var(Z_j^*|H_0)}}. \tag{C.5}$$

Thus, we need to determine how to calculate $E(Z_j^*|H_0)$, $Var(Z_j^*|H_0)$, $E(Z_j^*|H_a)$, and $Var(Z_j^*|H_a)$. These values depend on the distribution of Z_j (see Appendix A) under the null and alternative hypotheses.

One possible set of hypotheses, that take into account the assumption that transaction are identically distributed within cells, is:

H_0: $\mu_{1j} = \mu_{2j}$, $\sigma_{1j}^2 = \sigma_{2j}^2$

H_a: $\mu_{2j} = \mu_{1j} + \delta_j \cdot \sigma_{1j}$, $\sigma_{2j}^2 = \lambda_j \cdot \sigma_{1j}^2$ $\quad \delta_j > 0$, $\lambda_j \geq 1$ and $j = 1,\ldots,L$.

Under this null hypothesis, Z_j has a standard normal distribution within each cell j. In which case,

$$E(Z_j^*|H_0) = -\frac{1}{\sqrt{2\pi}}, \text{ and}$$

$$var(Z_j^*|H_0) = \frac{1}{2} - \frac{1}{2\pi}.$$

Under the alternative hypothesis, Z_j has a normal distribution with

$$E(Z_j|H_a) = m_j = \frac{-\delta_j}{\sqrt{\frac{1}{n_{1j}} + \frac{1}{n_{2j}}}}, \text{ and}$$

$$SE(Z_j|H_a) = se_j = \sqrt{\frac{\lambda_j n_{1j} + n_{2j}}{n_{1j} + n_{2j}}}$$

In general, the mean of a normal distribution truncated at 0 is

$$M(\mu,\sigma) = \int_{-\infty}^{0} \frac{x}{\sqrt{2\pi}\sigma} \exp(-\tfrac{1}{2}\left(\tfrac{x-\mu}{\sigma}\right)^2)dx,$$

and the variance is

$$V(\mu,\sigma) = \int_{-\infty}^{0} \frac{x^2}{\sqrt{2\pi}\sigma} \exp(-\tfrac{1}{2}\left(\tfrac{x-\mu}{\sigma}\right)^2)dx - M(\mu,\sigma)^2$$

It can be shown that

$$M(\mu,\sigma) = \mu\,\Phi(\tfrac{-\mu}{\sigma}) - \sigma\,\phi(\tfrac{-\mu}{\sigma})$$

and

$$V(\mu,\sigma) = (\mu^2 + \sigma^2)\Phi(\tfrac{-\mu}{\sigma}) - \mu\sigma\phi(\tfrac{-\mu}{\sigma}) - M(\mu,\sigma)^2$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function, and $\phi(\cdot)$ is the standard normal density function.

Using the above notation, and equation (C.5)~~(C.5)~~~~(C.5)~~~~(C.5)~~, we get the formula for the balancing critical of Z^T for the alternative hypothesis defined above.

$$c_B = \frac{\sum_j W_j M(m_j, se_j) - \sum_j W_j \frac{-1}{\sqrt{2\pi}}}{\sqrt{\sum_j W_j^2 V(m_j, se_j)} + \sqrt{\sum_j W_j^2 \left(\frac{1}{2} - \frac{1}{2\pi}\right)}}. \qquad (C.6)$$

This formula assumes that Z_j, is approximately normally distributed within cell j. When the cell sample sizes, n_{1j} and n_{2j}, are small this may not be true. It is possible to determine the cell mean and variance under the null hypothesis when the cell sample sizes are small. It is much more difficult to determine these values under the alternative hypothesis. Since the cell weight, W_j will also be small (see Appendix A) for a cell with small volume, the cell mean and variance will not contribute much to the weighted sum. Therefore, formula (C.6)~~(C.6)~~~~(C.6)~~~~(C.6)~~ should provide a reasonable approximation to the balancing critical value.

Determining the Parameters of the Alternative Hypothesis

In this appendix we have indexed the alternative hypothesis by two sets of parameters, λ_j and δ_j. While statistical science can be used to evaluate the impact of different choices of these parameters, there is not much that an appeal to statistical principles can offer in directing specific choices. Specific choices are best left to telephony experts. Still, it is possible to comment on some aspects of these choices:

- Parameter Choices for λ_j. The set of parameters λ_j index alternatives to the null hypothesis that arise because there might be greater unpredictability or variability in the delivery of service to a CLEC customer over that which would be achieved for an otherwise comparable ILEC customer. While concerns about differences in the variability of service are important, it turns out that the truncated Z testing which is being recommended here is relatively insensitive to all but very large values of the λ_j. Put another way, reasonable differences in the values chosen here could make very little difference in the balancing points chosen.

- Parameter Choices for δ_j. The set of parameters δ_j are much more important in the choice of the balancing point than was true for the λ_j. The reason for this is that they directly index differences in average service. The truncated Z test is very sensitive to any such differences; hence, even small disagreements among experts in the choice of the δ_j could be very important. Sample size matters here too. For example, setting all the δ_j to a single value — $\delta_j = \delta$ — might be fine for tests across individual CLECs where currently in Louisiana the CLEC customer bases are not too different. Using the same value of δ for the overall state testing does not seem sensible, however, since the state sample would be so much larger.

The bottom line here is that beyond a few general considerations, like those given above, a principled approach to the choice of the alternative hypotheses to guard against must come from elsewhere.